OPERATIONS RESEARCH

Linear Programming: The Graphical Method

Dr. *Krishan K. Pandey*

LP Problem Solution

An optimal, as well as feasible solution to an LP problem is obtained by choosing from several values of decision variables $x_1, x_2, \ldots, x_n$, the one set of values that satisfies the given set of constraints simultaneously and also provides the optimal (maximum or minimum) value of the given objective function.

LP problems with two variables, the entire set of feasible solutions can be displayed graphically by plotting linear constraints on a graph paper to best (optimal) solution. The technique used to identify the optimal solution is called the *graphical solution approach*.

This solution approach provides valuable understanding of how to solve LP problems involving more than two variables algebraically.

Graphical solution methods or approaches are:

- Extreme point solution method
- Iso-profit (cost) function line method used to find the optimal solution to an LP problem.

Important Definitions

Solution: The set of values of decision variables x_j (j = 1, 2, . . ., n) which satisfy the constraints of an LP problem is said to constitute solution to that LP problem.

Feasible solution: The set of values of decision variables x_j (j = 1, 2, . . ., n) which satisfy all the constraints and non-negativity conditions of an LP problem simultaneously is said to constitute the feasible solution to that LP problem.

Infeasible solution: The set of values of decision variables x_j (j = 1, 2, . . ., n) which do not satisfy all the constraints and non-negativity conditions of an LP problem simultaneously is said to constitute the infeasible solution to that LP problem.

Basic solution: For a set of m simultaneous equations in n variables (n > m), a solution obtained by setting (n – m) variables equal to zero and solving for remaining m equations in m variables is called a basic solution.

The (n – m) variables whose value did not appear in this solution are called non-basic variables and the remaining m variables are called basic variables.

Basic feasible solution: A feasible solution to an LP problem which is also the basic solution is called the basic feasible solution. That is, all basic variables assume non-negative values. Basic feasible solutions are of two types:

- Degenerate: A basic feasible solution is called degenerate if value of at least one basic variable is zero.
- Non-degenerate: A basic feasible solution is called non-degenerate if values all m basic variables are non-zero and positive.

Optimum basic feasible solution: A basic feasible solution which optimizes (maximizes or minimizes) the objective function value of the given LP problem is called an optimum basic feasible solution.

Unbounded solution: A solution which can increase or decrease the value of objective function of the LP problem indefinitely is called an unbounded solution.

Standard Results or Theorems

While obtaining the optimal solution to the LP problem by the graphical method, the statement of the following theorems is used.

- The collection of all feasible solutions to an LP problem constitutes a convex set whose extreme points correspond to the basic feasible solutions.
- There are a finite number of basic feasible solutions within the feasible solution space.
- If the convex set of the feasible solutions of the system of simultaneous equations: $\mathbf{Ax = b, x \geq 0}$, is a convex polyhedron, then at least one of the extreme points gives an optimal solution.
- If the optimal solution occurs at more than one extreme point, the value of the objective function will be the same for all convex combinations of these extreme points.

Extreme Point Solution Method

Coordinates of all corner (or extreme) points of the feasible region (space or area) are determined and then values of the objective function at these points are computed and compared because an optimal solution to any LP problem always lie at one of the corner (extreme) points of the feasible solution space.

Step 1: Develop LP model

State the given problem in the mathematical LP model.

Step 2: Plot constraints on graph paper and decide the feasible region

- Replace the inequality sign in each constraint by an equality sign.
- Draw these straight lines on the graph paper and decide each time the area of feasible solutions according to the inequality sign of the constraint. Shade the common portion of the graph that satisfies all the constraints simultaneously drawn so far.

- The final shaded area is called the *feasible region* (*or solution space*) of the given LP problem. Any point inside this region is called *feasible solution* and provides values of x_1 and x_2 that satisfy all constraints.

Step 3: Examine extreme points of the feasible solution space to find an optimal solution

- Determine the coordinates of each extreme point of the feasible solution space.
- Compute and compare the value of the objective function at each extreme point.
- Identify extreme point that gives optimal (max. or min.) value of the objective function.

Example 1: Use the graphical method to solve the following LP problem.

Maximize $Z = 15x_1 + 10x_2$

subject to the constraints

$$4\,x_1 + 6\,x_2 \leq 360$$

$$3\,x_1 + 0\,x_2 \leq 180$$

$$0\,x_1 + 5\,x_2 \leq 200$$

and $\quad x_1, x_2 \geq 0.$

Solution

- The given LP problem is already in mathematical form.
- We shall treat x_1 as the horizontal axis and x_2 as the vertical axis. Each constraint will be plotted on the graph by treating it as a linear equation and then appropriate inequality conditions will be used to mark the area of feasible solutions.

Consider the first constraint $4x_1 + 6x_2 \leq 360$. Treat it as the equation, $4x_1 + 6x_2 = 360$. The easiest way to plot this line is to find any two points that satisfy the equation, then drawing a straight line through them. The two points are generally the points at which the line intersects the x_1 and x_2 axes. For example, when $x_1 = 0$ we get $6x_2 = 360$ or $x_2 = 60$. Similarly when $x_2 = 0$, $4x_1 = 360$, $x_1 = 90$.

These two points are then connected by a straight line. $4\ x_1 + 6\ x_2 \leq 360$. Any point above the constraint line violates the inequality condition. But any point below the line does not violate the constraint. Thus, the inequality and non-negativity condition can only be satisfied by the shaded area (feasible region) as shown in the figure.

Similarly, the constraints $3\ x_1 \leq 180$ and $5\ x_2 \leq 200$ are also plotted on the graph and are indicated by the shaded area in the figure.

Since all constraints have been graphed, the area which is bounded by all the constraints lines including all the boundary points is called the *feasible region* (*or solution space*). The feasible region is shown by the shaded area OABCD in the figure.

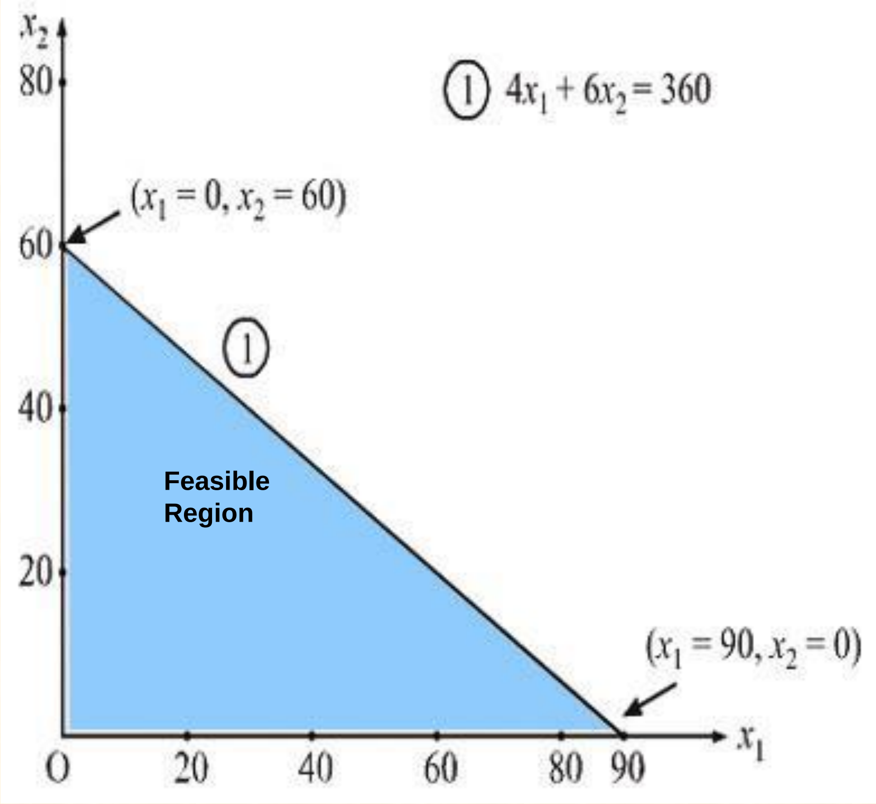

Fig. 1 Graphical Solution of LP Problem

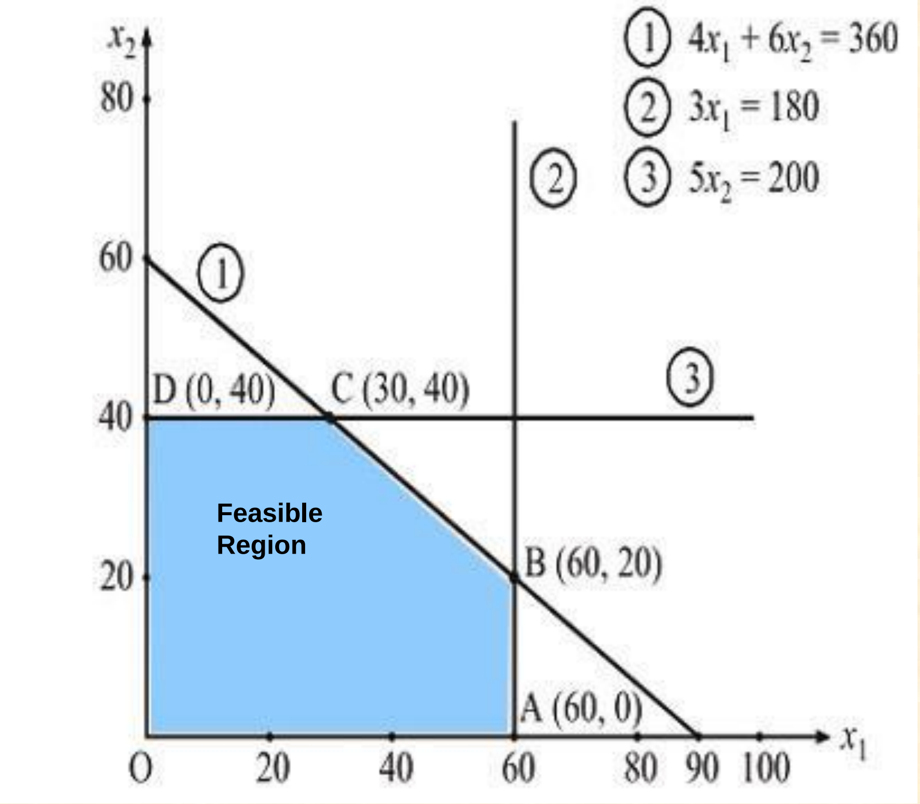

Fig. 1 Graphical Solution of LP Problem

3. i) Since the optimal value of the objective function occurs at one of the extreme points of the feasible region, it is necessary to determine their coordinates. The coordinates of extreme points of the feasible region are: O = (0, 0), A = (60, 0), B = (60, 20), C = (30, 40), D = (0, 40).

ii) Evaluate objective function value at each extreme point of the feasible region :

Extreme Point	***Coordinates*** (x_1, x_2)	***Objective Function Value*** $Z = 15x_1 + 10x_2$
O	(0, 0)	15(0) + 10(0) = 0
A	(60, 0)	15(60) + 10(0) = 900
B	(60, 20)	15(60) + 10(20) = 1,100
C	(30, 40)	5(30) + 10(40) = 850
D	(0, 40)	15(0) + 10(40) = 400

iii) Since maximum value of Z = 1,100 is achieved at the point extreme B (60, 20). Hence the optimal solution to the given LP problem is: $x_1 = 60$, $x_2 = 20$ and Max Z = 1,100.

Example 2: Use the graphical method to solve the following LP problem.

Maximize $Z = 2x_1 + x_2$

subject to the constraints

$$x_1 + 2x_2 \leq 10$$

$$x_1 + x_2 \leq 6$$

$$x_1 - x_2 \leq 2$$

$$x_1 - 2x_2 \leq 1$$

and $\quad x_1, x_2 \leq 0$

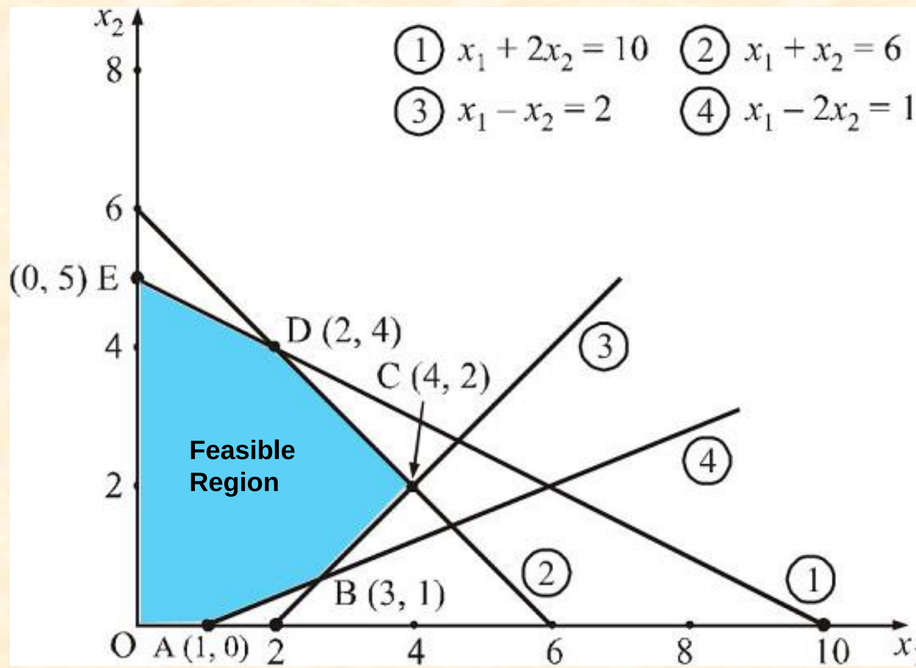

Fig. 3 Graphical Solution of LP Problem -2

Extreme Point	*Coordinates* (x_1, x_2)	*Objective Function Value* $Z = 2x_1 + x_2$
O	(0, 0)	2(0) + 1(0) = 0
A	(1, 0)	2(1) + 1(0) = 2
B	(3, 1)	2(3) + 1(1) = 7
C	(4, 2)	2(4) + 1(2) = 10
D	(2, 4)	2(2) + 1(4) = 8
E	(0, 5)	2(0) + 1(5) = 5

The maximum value of the objective function $Z = 10$ occurs at the extreme point (4, 2). Hence, the optimal solution to the given LP problem is: $x_1 = 4$, $x_2 = 2$ and Max $Z = 10$.

Example 3: Use the graphical method to solve the LP problem.

Minimize $Z = 3x_1 + 2x_2$

subject to the constraints

$$5x_1 + x_2 \geq 10$$

$$x_1 + x_2 \geq 6$$

$$x_1 + 4x_2 \geq 12$$

and $\quad x_1, x_2 \geq 0$

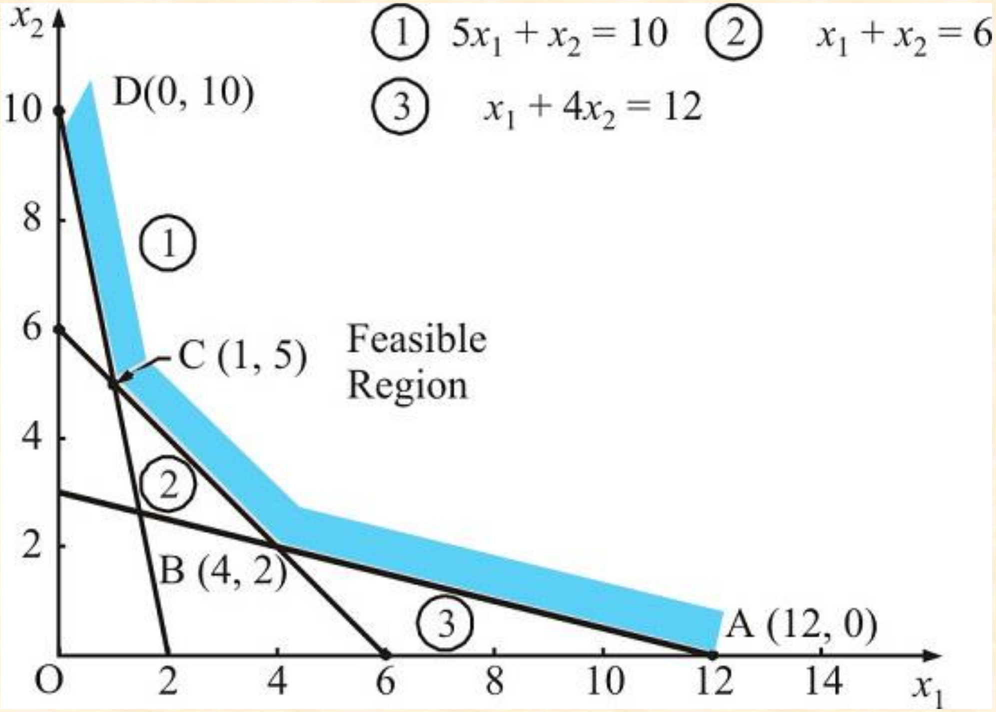

Fig. 4 Graphical Solution of LP Problem -3

Extreme Point	*Coordinates* (x_1, x_2)	*Objective Function Value* $Z = 3x_1 + 2x_2$
A	(12, 0)	3(12) + 2(0) = 36
B	(4, 2)	3(4) + 2(2) = 16
C	(1, 5)	3(1) + 2(5) = 13
D	(0, 10)	3(0) + 2(10) = 20

The minimum value of the objective function $Z = 13$ occurs at the extreme point C (1, 5). Hence, the optimal solution to the given LP problem is: $x_1 = 1$, $x_2 = 5$, and Min $Z = 13$.

Example 4: Use the graphical method to solve the LP problem.

Minimize $Z = -x_1 + 2x_2$

subject to the constraints

$$-x_1 + 3x_2 \leq 10$$

$$x_1 + x_2 \leq 6$$

$$x_1 - x_2 \leq 2$$

and $x_1, x_2 \geq 0.$

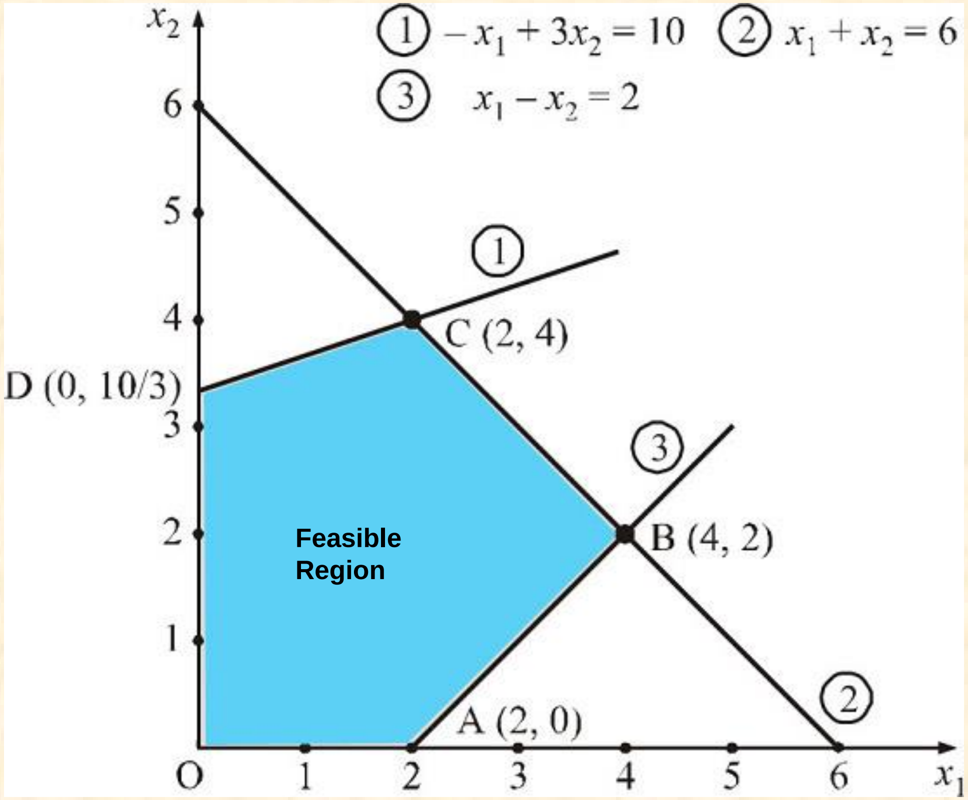

Fig. 5 Graphical Solution of LP Problem -4

Extreme Point	*Coordinates* (x_1, x_2)	*Objective Function Value* $Z = -x_1 + 2x_2$
O	(0, 0)	– 1(0) + 2(0) = 0
A	(2, 0)	– 1(2) + 2(0) = – 2
B	(4, 2)	– 1(4) + 2(2) = 0
C	(2, 4)	– 1(2) + 2(4) = 6
D	(0, 10/3)	– 1(0) + 2(10/3) = 20/3

The minimum value of the objective function $Z = -2$ occurs at the extreme point A (2, 0). Hence, the optimal solution to the given LP problem is: $x_1 = 2$, $x_2 = 0$ and Min $Z = -2$.

Example 5: Use the graphical method to solve the LP problem.

Maximize $Z = 2x_1 + 3x_2$

subject to the constraints

$$\begin{aligned} x_1 + x_2 &\leq 30 \\ x_2 &\geq 3 \\ 0 \leq x_2 &\leq 12 \\ 0 \leq x_1 &\leq 20 \\ x_1 - x_2 &\geq 0 \end{aligned}$$

and $x_1, x_2 \geq 0$

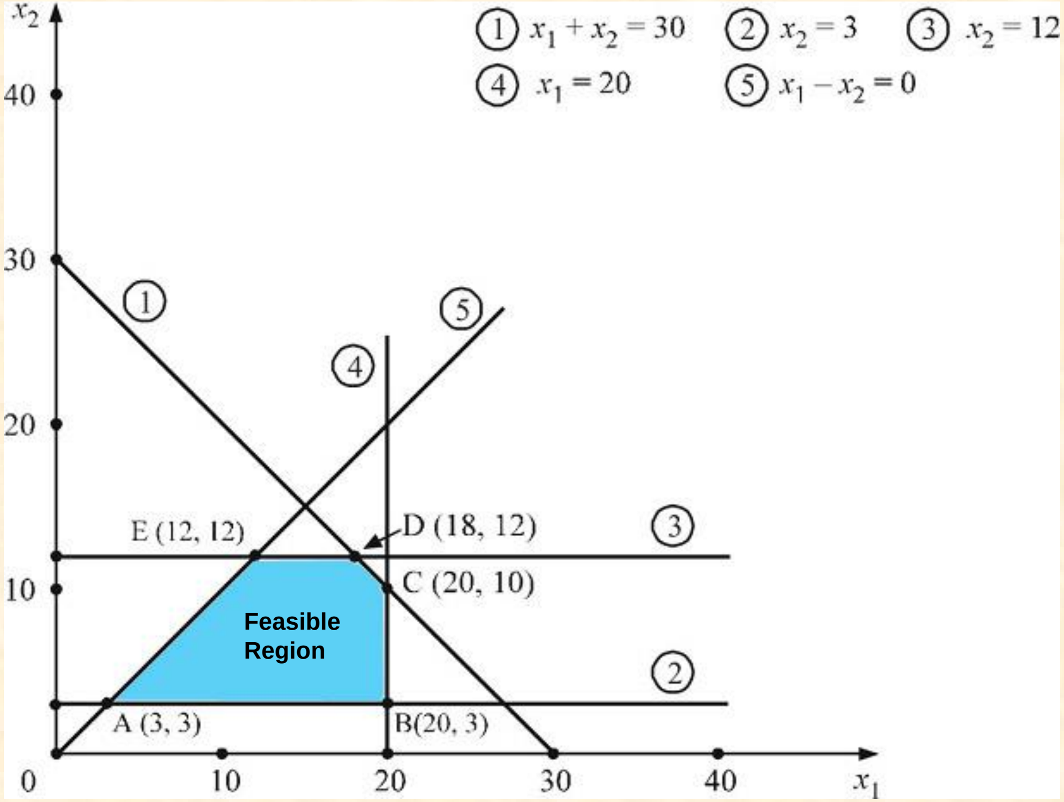

Fig. 6 Graphical Solution of LP Problem -5

Extreme Point	*Coordinates* (x_1, x_2)	*Objective Function Value* $Z = 2x_1 + 3x_2$
A	(3, 3)	2(3) + 3(3) = 15
B	(20, 3)	2(20) + 3(3) = 49
C	(20, 10)	2(20) + 3(10) = 70
D	(18, 12)	2(18) + 3(12) = 72
E	(12, 12)	2(12) + 3(12) = 60

The maximum value of the objective function $Z = 72$ occurs at the extreme point D (18, 12). Hence, the optimal solution to the given LP problem is: $x_1 = 18$, $x_2 = 12$ and Max $Z = 72$.

Example 6: A firm makes two products X and Y, and has a total production capacity of 9 tones per day, X and Y requiring the same production capacity. The firm has a permanent contract to supply at least 2 tones of X and at least 3 tones of Y per day to another company. Each tone of X requires 20 machine hours of production time and each tone of Y requires 50 machine hours of production time. The daily maximum possible number of machine hours is 360. All the firm's output can be sold, and the profit made is Rs. 80 per tone of X and Rs. 120 per tone of Y. It is required to determine the production schedule for maximum profit and to calculate this profit.

Extreme Point	*Coordinates* (x_1, x_2)	*Objective Function Value* $Z = 80x_1 + 120x_2$
A	(2, 3)	80(2) + 120(3) = 520
B	(6, 3)	80(6) + 120(3) = 840
C	(3, 6)	80(3) + 120(6) = 960
D	(2, 6.4)	80(2) + 120(6.4) = 928

The maximum value of the objective function $Z = 960$ occurs at the extreme point C (3, 6). Hence the company should produce, $x_1 = 3$ tones of product X and $x_2 = 6$ tones of product Y in order to yield a maximum profit of Rs. 960.

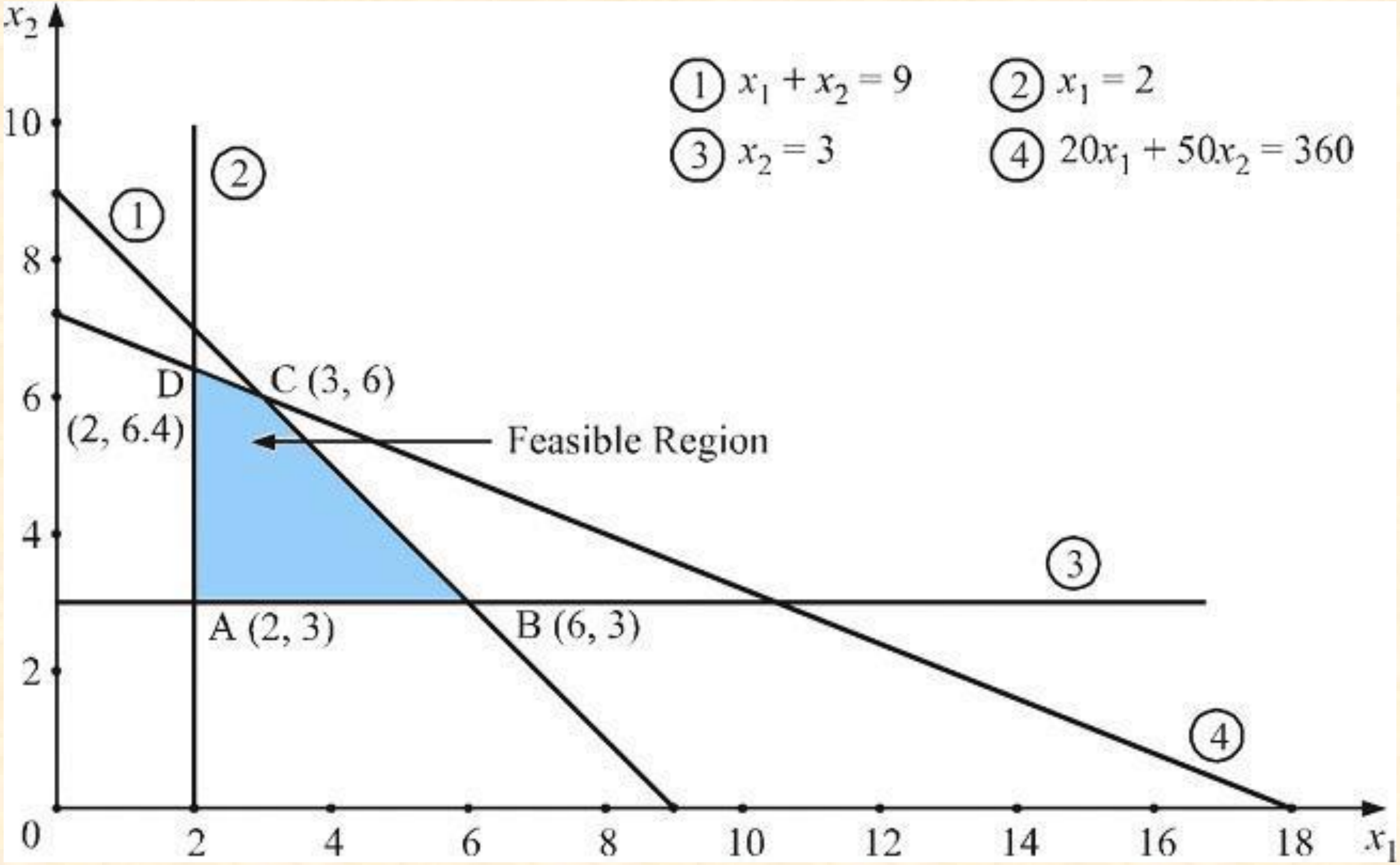

Fig. 7 Graphical Solution of LP Problem -6

Example 7: A manufacturer produces two different models: X and Y, of the same product. Model X makes a contribution of Rs 50 per unit and model Y, Rs 30 per unit towards total profit. Raw materials r_1 and r_2 are required for production. At least 18 kg of r_1 and 12 kg of r_2 must be used daily. Also at most 34 hours of labour are to be utilized.

A quantity of 2 kg of r_1 is needed for model X and 1 kg of r_1 for model Y. For each of X and Y, 1 kg of r_2 is required.

It takes 3 hours to manufacture model X and 2 hours to manufacture model Y.

How many units of each model should be produced to maximize the profit?

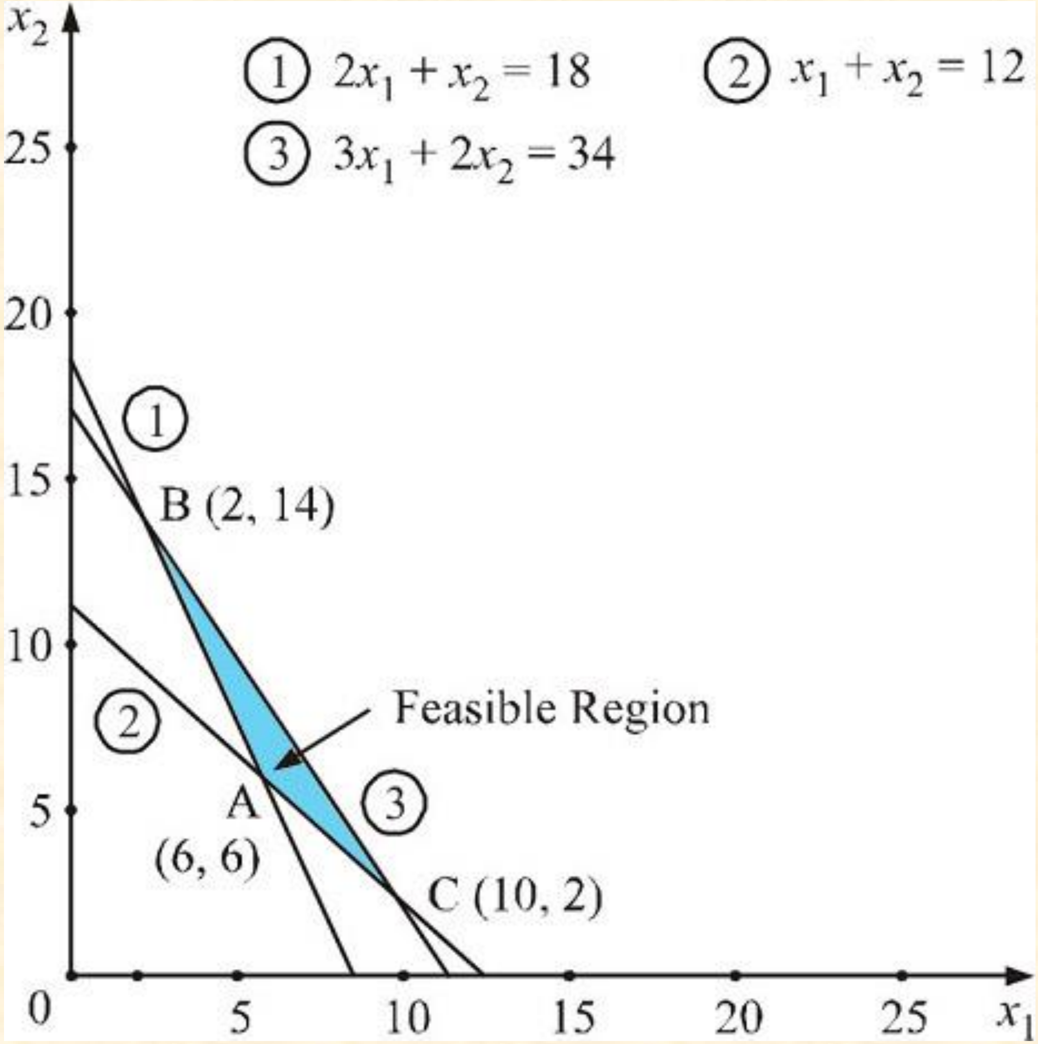

Fig. 8 Graphical Solution of LP Problem - 7

Extreme Point	*Coordinates* (x_1, x_2)	*Objective Function Value* $Z = 50x_1 + 30x_2$
A	(6, 6)	50(6) + 30(6) = 480
B	(2, 14)	50(2) + 30(14) = 520
C	(10, 2)	50(10) + 30(2) = 560

Since the maximum value of Z = 560 occurs at the point C (10, 2), the manufacturer should produce x_1 = 10 units of model X and x_2 = 2 units of Y to yield a maximum profit of Rs. 560.